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Hypoellipticity for certain systems of complex vector fields

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Abstract

In this work we study the smooth and Gevrey hypoellipticity for systems of n real-analytic, complex vector fields in $n + 1$ variables. Our results are stated in terms of the properties of a first integral of the system. In particular, we give a partial answer to a conjecture stated by F. Treves in 1981.

MSC: primary 35F05

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1. Introduction

In 1981, François Treves stated a conjecture for the validity of the smooth hypoellipticity for certain systems defined by complex, real-analytic vector fields. More precisely, given n ($n \geq 1$) linearly independent, real-analytic, pairwise commuting vector fields L_j, defined in an open neighborhood of the origin in $\mathbb{R}^{n+1}$, the overdetermined system $\mathrm{L}_j u = f$ would be hypoelliptic in a neighborhood of the origin if and only if any first integral Z of the system, that is, any smooth solution to the homogeneous system $\mathrm{L}Z = 0$, with $\mathrm{d}Z \neq 0$, is everywhere open. With

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the exception of the case when the system is in the so called tube form, for which Maire [6] has proved the validity of the conjecture, the general case is still open.

We must point out that the case $n = 1$ is completely known. In this case it is well known that a nonvanishing, real analytic complex vector field in the plane is smooth (and analytic) hypoelliptic if and only if it satisfies conditions (P) and (Q) (cf. Treves [10], Theorem 24.7.2), properties that can be proved to be equivalent to the openness of any first integral.

In this work we assume $n \geq 2$ and show that the conjecture of Treves holds if we impose an additional condition on the first integral Z (cf. Definition 2.2 below). Indeed we show more: Gevrey hypoellipticity, and even hypoellipticity for a large class of Denjoy-Carleman sequences also hold (cf Theorems 2.3, 2.4 and 7.1 below). The Gevrey hypoellipticity for systems in tube form has already been established in [3].

Our proofs make use of the operators with complex phase introduced in Treves [8], but with an important improvement: the use of almost holomorphic extensions. Although this would not be crucial in the smooth or Gevrey cases, this approach allows us to obtain the result for more general Denjoy-Carleman sequences, since it avoids the use of more precise cut off functions (notice that we are including here some quasi-analytic classes). The existence of such precise almost holomorphic extensions is due to A. Victor da Silva in a work in collaboration with the first author [1].

Finally we point out the progress that has been made in the study of the subellipticity for such class of systems, also in the tube form [4], [5].

2. Preliminaries - statement of the results

We shall consider a corank one, real-analytic, locally integrable structure defined in an open neighborhood Ω of the origin in $\mathbb{R}^{n+1}$, that is, a real-analytic line subbundle $\mathrm{T}' \subset \mathbb{C} \otimes T^\star\Omega$ which is (locally) generated by the differential of a complex valued real-analytic function. A *(classical) solution* for T' is a C^1 function defined in an open subset of Ω whose differential is a section of T'. Real-analytic solutions with non vanishing differentials will be referred to as *first integrals of* T'. We can extend the notion of solution to distributions: a *distribution solution* in some open subset U of Ω is a distribution on U which is locally annihilated by any section of the orthogonal bundle of T'.

It is well known that to the structure T' we can associate a differential complex, whose first term is given by the composition

$$\mathrm{d}_0' : C^\infty(\Omega) \to C^\infty(\Omega; \mathbb{C} \otimes T^\star\Omega) \to C^\infty(\Omega; (\mathbb{C} \otimes T^\star\Omega)/\mathrm{T}'),$$

where the first arrow denotes the exterior derivative, and the second stands for the map induced by the projection. In particular, a solution for T' is simply a solution of the homogeneous equation

$$\mathrm{d}_0' u = 0. \tag{1}$$

Our goal will be to study the smooth and Gevrey regularity (hypoellipticity) for the solutions of the non homogeneous equation

$$\mathrm{d}_0' u = f. \tag{2}$$

In order to state our results we must introduce some definitions. We recall that T' is *hypocomplex* in Ω if every point $p \in \Omega$ satisfies the following property: *given any distribution solution u for T' defined in a neighborhood of p and given any first integral Z of T' near p there is a holomorphic function h defined in an open neighborhood of $Z(p)$ in the complex plane such that $u = h \circ Z$.* In particular, every distribution solution for T' is indeed real-analytic.

Hypocomplex structures of corank one are completely characterized by the following result due to F. Treves [9], Corollary 3.5.3:

Proposition 2.1. *The corank one structure T' as above is hypocomplex in Ω if given any point p there is a first integral Z defined near p which is open at p.*

Our next definition is a bit more technical:

Definition 2.2. We say that T' satisfies Condition (N) (at the origin) if there is a first integral Z defined in some open neighborhood $\Omega' \subset \Omega$ of the origin satisfying $\mathrm{d}(\operatorname{Re} Z)(0) \neq 0$ and, for some constant $C > 0$,

$$|\mathrm{d}(\operatorname{Im} Z)| \leq C|\mathrm{d}Z \wedge \mathrm{d}\bar{Z}| \quad \text{in } \Omega'. \tag{3}$$

Remark. Here the pointwise norms are taken relative to any trivialization of the bundle $\mathbb{C} \otimes T^*\Omega'$. Changes of variables result in uniformly equivalent pointwise norms. Thus condition (N) is, indeed, an invariant condition.

Recall that the zero set of $\mathrm{d}Z \wedge \mathrm{d}\bar{Z}$ is precisely the base projection Σ of the characteristic set of T', that is, the base projection of the set $\mathrm{T}' \cap T^*\Omega$, and hence if T' is elliptic then it satisfies condition (N). In particular condition (3) implies that $\mathrm{d}(\operatorname{Im} Z)$ must vanish on Σ. Conversely, if $\mathrm{d}(\operatorname{Im} Z)$ vanishes on Σ then we obtain, via Lojasiewicz inequality, a weaker version of (3), in which the left hand side is raised to some possible large power; condition (N) means that such power can be taken equal to one.

We are now ready to state the results that will be proved in the next sections:

Theorem 2.3. *Let T' define a real-analytic, corank one locally integrable structure on an open neighborhood Ω of the origin in $\mathbb{R}^{n+1}$. If T' is hypocomplex in Ω and satisfies condition (N) then given $U \subset \Omega$, an open neighborhood of the origin, there is another such neighborhood $V \subset U$ for which the following is true: if u is a distribution in U which satisfies (2) and if $f \in C^\infty(U; (\mathbb{C} \otimes T^\star)/\mathrm{T}')$ then u is smooth on V.*

Theorem 2.4. *Let T' define a real-analytic, corank one locally integrable structure on an open neighborhood Ω of the origin in $\mathbb{R}^{n+1}$. If T' is hypocomplex in Ω and satisfies condition (N) then given $U \subset \Omega$, an open neighborhood of the origin, there is another such neighborhood $V \subset U$ for which the following is true: if u is a distribution solution of (2) in U and if the coefficients of f are Gevrey functions of order s ($s \geq 1$, fixed) in U then u is Gevrey of order s on V.*

We remark that Theorem 2.4 is valid when $s = 1$ without assuming property (N). This follows from the fact that under the assumption of hypocomplexity, as already mentioned, every solution of (1) is real-analytic and, by the Cauchy-Kowalevsky Theorem, equation (2) has local real-analytic solutions if f has analytic coefficients. Thus, in what follows, we shall restrict to the case $s > 1$.

3. Local coordinates and generators

Let the first integral Z satisfy the properties in condition (N). There is no loss of generality assuming that $Z(0) = 0$. We take $\operatorname{Re} Z$ as a coordinate x and complete it to a coordinate system $(x, t_1, \ldots, t_n)$ in a neighborhood of the origin. More precisely, we shall assume that $U = I \times B$, where I (resp. B) is an open interval in $\mathbb{R}$ (resp. open ball), both centered at the respective origins. Here we assume $U \subset \Omega'$, where the latter was introduced in the definition of condition (N). We then write, for $(x, t) \in I \times B$,

$$Z(x, t) = x + i\phi(x, t),$$

where the function ϕ is real-valued, real-analytic and satisfies $\phi(0, 0) = 0$. Hypocomplexity now reads

$$\forall x \in I, \text{ the map } B \ni t \mapsto \phi(x, t) \text{ is open in } B. \tag{4}$$

Next we have $\mathrm{d}(\operatorname{Im} Z) = \mathrm{d}\phi$ and $\mathrm{d}Z \wedge \mathrm{d}\bar{Z} = -2i\,\mathrm{d}x \wedge \mathrm{d}\phi = -2i\,\mathrm{d}x \wedge \mathrm{d}_t\phi$ in U and thus

$$|\mathrm{d}(\operatorname{Im} Z)| \simeq |\phi_x| + |\mathrm{d}_t\phi|, \quad |\mathrm{d}Z \wedge \mathrm{d}\bar{Z}| \simeq |\mathrm{d}_t\phi|.$$

Consequently, (3) is equivalent to

$$|\phi_x| \leq C_1|\mathrm{d}_t\phi|, \quad (x, t) \in U. \tag{5}$$

After contracting U around the origin, by the Lojasiewicz inequality (see [2]) there are $1/2 \leq \theta \leq 1$ and $C_2 > 0$ such that

$$|\phi(x, t)|^\theta \leq C_2|\mathrm{d}\phi(x, t)| \quad (x, t) \in U,$$

and consequently, by (5), we obtain, for a new constant $A > 0$,

$$|\phi(x, t)|^\theta \leq A|\mathrm{d}_t\phi(x, t)|, \quad (x, t) \in U. \tag{6}$$

In what follows we shall restrict ourselves to the non elliptic case, and then we assume $1/2 \leq \theta < 1$.

We now introduce the vector fields

$$\mathrm{M} = (1 + i\phi_x(x, t))^{-1}\frac{\partial}{\partial x} \tag{7}$$

and

$$\mathrm{L}_j = \frac{\partial}{\partial t_j} - i\frac{\partial \phi}{\partial t_j}\mathrm{M}, \quad j = 1, \ldots, n. \tag{8}$$

It is easily seen that $\mathrm{M}, \mathrm{L}_1, \ldots, \mathrm{L}_n$ span $(\mathbb{C} \otimes T\mathbb{R}^{n+1})|_U$ and are pairwise commuting; indeed, $\{\mathrm{M}, \mathrm{L}_1, \ldots, \mathrm{L}_n\}$ is the dual basis of $\{\mathrm{d}Z, \mathrm{d}t_1, \ldots, \mathrm{d}t_n\}$. Moreover, the one forms $\mathrm{d}t_1, \ldots, \mathrm{d}t_n$ span $\left\{(\mathbb{C} \otimes T^*\Omega)/\mathrm{T}'\right\}|_U$ and, since for a C^1-function v we have

$$\mathrm{d}v = (\mathrm{M}v)\mathrm{d}Z + \sum_{j=1}^{n} (\mathrm{L}_j v)\mathrm{d}t_j,$$

it follows that

$$\mathrm{d}_0' v = \sum_{j=1}^{n} (\mathrm{L}_j v)\mathrm{d}t_j. \tag{9}$$

Summing up, we have reduced our regularity problem to the study of the system of first order PDE

$$\mathrm{L}_j u = f_j, \quad j = 1, \ldots, n,$$

where the real-analytic function $\phi : U \to \mathbb{R}$ satisfies (4) and (6).

We recall that we are denoting by Σ the projection over the base of the characteristic set of T'. It can be described, in U, as

$$\Sigma = \{(x,t) \in U : \mathrm{d}_t\phi(x,t) = 0\},$$

and hypocomplexity implies that

$$\Sigma_x = \{t \in B : \mathrm{d}_t\phi(x,t) = 0\},$$

is a proper real-analytic set of B if $x \in I$.

Proposition 3.1. *Assume that the real-valued, real-analytic map ϕ satisfies* (4) *and* (6) *and let $I \subset \mathbb{R}$, $B \subset \mathbb{R}^n$ have the meaning as above. Then there are two families of continuous, piecewise real-analytic curves $\gamma^{\pm}_{x,t} : [0, \delta^{\pm}_0(x,t)] \longrightarrow \overline{B}$, where $x \in I, t \in B \setminus \Sigma_x$, such that $\gamma^{\pm}_{x,t}(\delta^{\pm}_0(x,t)) \in \partial B$, satisfying the following properties:*

(1) *For every $B_1 \Subset B$ open ball centered at the origin there exist $\delta^{\pm} > 0$ such that $\delta^{\pm}_0(x,t) \geq \delta^{\pm}$, for every $x \in I$, and $t \in B_1 \setminus \Sigma_x$;*
(2) *The length of the curves $\gamma^{\pm}_{x,t}$ is uniformly bounded for every $x \in I$, and $t \in B \setminus \Sigma_x$;*
(3) *If $x \in I$, $t \in B_1 \setminus \Sigma_x$ and $0 \leq \tau \leq \delta^{+}_0(x,t)$, then*

$$\phi(x,t) - \phi(x, \gamma^{+}_{x,t}(\tau)) \geq \frac{1}{2}\left((1-\theta)(8A)^{-1}\right)^{\frac{1}{1-\theta}} \tau^{\frac{1}{1-\theta}}$$

(4) *If $x \in I$, $t \in B_1 \setminus \Sigma_x$ and $0 \leq \tau \leq \delta^{-}_0(x,t)$, then*

$$\phi(x, \gamma^{-}_{x,t}(\tau)) - \phi(x,t) \geq \frac{1}{2}\left((1-\theta)(8A)^{-1}\right)^{\frac{1}{1-\theta}} \tau^{\frac{1}{1-\theta}}$$

The proof of this result follows from a standard argument due originally to Maire [6] and which was also presented in [3]. We leave the details to the interested reader.

4. Study of some integral operators

Since we are assuming that T' is not elliptic at the origin, condition (N) implies that $\phi_x(0,0) = 0$ and hence, after contracting U around the origin if necessary, we can assume that

$$\sup_{(x,t)\in U} |\phi_x(x,t)| \leq \frac{1}{4}. \tag{10}$$

Let $\chi \in C_c^\infty(I)$ and $u \in C^1(U)$. For every $\varepsilon > 0$ set

$$u_\varepsilon(x,t) = \frac{1}{2\pi}\int\limits_{\mathbb{R}}\int\limits_{I} u(y,t)\chi(y)e^{i\xi(Z(x,t)-Z(y,t))-\varepsilon|\xi|^2} Z_y(y,t)\mathrm{d}y\mathrm{d}\xi.$$

Thanks to (10) it is well known that $u_\varepsilon \to \chi u$, when $\varepsilon \to 0^+$, in the C^1-topology.

We decompose $u_\varepsilon = u_\varepsilon^+ + u_\varepsilon^-$, where in the first (resp. second) term the ξ-integral is performed over $[0,\infty[$ (resp. $]-\infty,0]$). We shall deal with the first term, the analysis of the other term being analogous.

Thus, let us write

$$u_\varepsilon^+(x,t) = \frac{1}{2\pi}\int\limits_0^\infty\int\limits_I e^{i\xi(Z(x,t)-Z(y,t))-\varepsilon\xi^2}\chi(y)u(y,t)Z_y(y,t)\mathrm{d}y\mathrm{d}\xi,$$

for every $(x,t) \in U$. For $k \in \mathbb{Z}_+$ we have

$$\mathrm{M}^k u_\varepsilon^+(x,t) = \frac{1}{2\pi}\int\limits_0^\infty\int\limits_I (i\xi)^k e^{i\xi(Z(x,t)-Z(y,t))-\varepsilon\xi^2}\chi(y)u(y,t)Z_y(y,t)\mathrm{d}y\mathrm{d}\xi.$$

Now let $x \in I$ and $t \in B_1 \setminus \Sigma_x$ be fixed and, for simplicity, we let γ denote $\gamma_{x,t}^+$. We note that

$$\begin{aligned}
&\mathrm{d}_s\left\{\frac{1}{2\pi}\int\limits_0^\infty\int\limits_I (i\xi)^k e^{i\xi(Z(x,t)-Z(y,s))-\varepsilon\xi^2}\chi(y)u(y,s)Z_y(y,s)\mathrm{d}y\mathrm{d}\xi\right\} = \\
&\qquad = \frac{1}{2\pi}\sum_{j=1}^n\left\{\frac{\partial}{\partial s_j}\int\limits_0^\infty\int\limits_I (i\xi)^k e^{i\xi(Z(x,t)-Z(y,s))-\varepsilon\xi^2}\chi(y)u(y,s)Z_y(y,s)\mathrm{d}y\mathrm{d}\xi\right\}\mathrm{d}s_j \\
&\qquad \overset{(*)}{=} \frac{1}{2\pi}\sum_{j=1}^n\left\{\int\limits_0^\infty\int\limits_I (i\xi)^k e^{i\xi(Z(x,t)-Z(y,s))-\varepsilon\xi^2}\mathrm{L}_j\big[\chi(y)u(y,s)\big]Z_y(y,s)\mathrm{d}y\mathrm{d}\xi\right\}\mathrm{d}s_j
\end{aligned}$$

$$= \frac{1}{2\pi}\sum_{j=1}^{n}\left\{\int_0^\infty\int_I (i\xi)^k e^{i\xi(Z(x,t)-Z(y,s))-\varepsilon\xi^2}\big[\mathrm{L}_j u(y,s)\big]\chi(y)Z_y(y,s)\mathrm{d}y\mathrm{d}\xi\right\}\mathrm{d}s_j$$

$$+ \frac{1}{2\pi}\sum_{j=1}^{n}\left\{\int_0^\infty\int_I (i\xi)^k e^{i\xi(Z(x,t)-Z(y,s))-\varepsilon\xi^2}\big[\mathrm{L}_j \chi(y)\big]u(y,s)Z_y(y,s)\mathrm{d}y\mathrm{d}\xi\right\}\mathrm{d}s_j,$$

where $(*)$ follows from standard commutation formulas (cf. [9], Chapter IX). Thus we have the following identity, valid for $x \in I$ and $t \in B_1 \setminus \Sigma_x$:

$$\mathrm{M}^k u_\varepsilon^+(x,t) =$$

$$= \underbrace{\frac{1}{2\pi}\int_0^\infty\int_I (i\xi)^k e^{i\xi\cdot(Z(x,t)-Z(y,t_*))-\varepsilon|\xi|^2}\chi(y)u(y,t_*)Z_y(y,t_*)\mathrm{d}y\mathrm{d}\xi}_{\doteq u^+_{1,k,\varepsilon}(x,t)} +$$

$$\underbrace{-\frac{1}{2\pi}\int_\gamma\sum_{j=1}^{n}\mathrm{d}s_j\left\{\int_0^\infty\int_I (i\xi)^k e^{i\xi\cdot(Z(x,t)-Z(y,s))-\varepsilon|\xi|^2}\big[\mathrm{L}_j u(y,s)\big]\chi(y)Z_y(y,s)\mathrm{d}y\mathrm{d}\xi\right\}}_{\doteq u^+_{2,k,\varepsilon}(x,t)}$$

$$\underbrace{-\frac{1}{(2\pi)}\int_\gamma\sum_{j=1}^{n}\mathrm{d}s_j\left\{\int_0^\infty\int_I (i\xi)^k e^{i\xi\cdot(Z(x,t)-Z(y,s))-\varepsilon|\xi|^2}\big[\mathrm{L}_j \chi(y)\big]u(y,s)Z_y(y,s)\mathrm{d}y\mathrm{d}\xi\right\}}_{\doteq u^+_{3,k,\varepsilon}(x,t)}.$$

We can estimate $u^+_{1,k,\varepsilon}(x,t)$ and $u^+_{3,k,\varepsilon}(x,t)$ without further assumptions on u. Indeed, let us take and open interval $J \subset\subset I$ also centered at the origin, and assume that χ satisfies $0 \le \chi \le 1$, $\chi \equiv 1$ on J. We notice that

$$\begin{aligned}
Z(x,t) - Z(y,s) &= x - y + i\big(\phi(x,t) - \phi(y,s)\big)\\
&= i\big(\phi(x,t) - \phi(x,s)\big) + i\big(\phi(x,s) - \phi(y,s)\big) + (x-y)\\
&= i\big(\phi(x,t) - \phi(x,s)\big) + (x-y)\big[1 + i\phi_1(x,y,s)\big],
\end{aligned}$$

where $\phi_1(x,y,s) = \int_0^1 \phi_x(y+\sigma(x-y),s)\mathrm{d}\sigma$. In view of (10), we have that $|\phi_1| \le 1/4$. On each integral that defines (I) and (II) we shall perform the following change in the ξ-integration:

$$\xi \mapsto \zeta \doteq \xi + \frac{i}{2}\frac{(x-y)}{|x-y|}\xi.$$

Then the real part of the exponents becomes

$$
\begin{aligned}
\mathrm{Re}\left\{i\zeta\left[Z(x,t)-Z(y,s)\right]-\varepsilon\zeta^2\right\} &= -\mathrm{Im}\left\{\zeta\left[Z(x,t)-Z(y,s)\right]\right\}-\varepsilon\mathrm{Re}\,\zeta^2 \\
&= -\xi(\phi(x,t)-\phi(x,s))-\xi(x-y)\phi_1(x,y,s) \\
&\quad -\frac{1}{2}|x-y|\xi-\frac{3\varepsilon}{4}\xi^2 \\
&\le -\frac{1}{2}\left((1-\theta)(8A)^{-1}\right)^{\frac{1}{1-\theta}}\tau^{\frac{1}{1-\theta}}\xi-\frac{3\varepsilon}{4}\xi^2-\frac{1}{4}|x-y|\xi,
\end{aligned} \tag{11}
$$

where we used that $s=\gamma(\tau)$, for some $0\le\tau\le\delta_0$, and that $\xi\ge 0$. Hence in (I) we obtain

$$
\mathrm{Re}\left\{i\zeta\left[Z(x,t)-Z(y,s)\right]-\varepsilon\zeta^2\right\}\le -\frac{1}{2}\left((1-\theta)(8A)^{-1}\right)^{\frac{1}{1-\theta}}\delta_+^{\frac{1}{1-\theta}}\xi \doteq -c\,\xi,
$$

where $c>0$, and so we can estimate

$$
|u^+_{1,k,\varepsilon}(x,t)|\le \mathrm{Const}\cdot\int_0^\infty e^{-c\xi}\left(\frac{3}{2}\xi\right)^k \mathrm{d}\xi \le C^{k+1}k!,\quad x\in I,\ t\in B_1\setminus\Sigma_x, \tag{12}
$$

for some constant $C>0$. On the second integral we notice that $\operatorname{supp}\mathrm{L}_j\chi\subset I\setminus I'$, for $j=1,\dots,n$. Hence we restrict x to the interval $J/2$, we have that if $y\in\operatorname{supp}\mathrm{L}_j\chi$ then $|x-y|\ge d>0$, therefore we can estimate it by

$$
|u^+_{3,k,\varepsilon}(x,t)|\le \mathrm{Const}\cdot\mathrm{length}\,(\gamma^+_{x,t})\cdot\int_0^\infty e^{-d\xi/4}\left(\frac{3}{2}\xi\right)^k\le C^{k+1}k!,\quad x\in J/2,\ t\in B_1\setminus\Sigma_x, \tag{13}
$$

for some positive constant $C>0$, where in the last inequality we have used a uniform bound for the lengths of $\gamma^+_{x,t}$. Summing up we have proved:

Proposition 4.1. *For each k we can estimate*

$$
|\mathrm{M}^k u^+_\varepsilon(x,t)|\le |u^+_{2,k,\varepsilon}(x,t)|+C^{k+1}k!,\quad x\in J/2,\ t\in B_1,
$$

where C is independent of k and $\varepsilon>0$.

Indeed, Σ_x being a proper analytic set in B_1 it has Lebesgue measure equal to zero. Proposition 4.1 is the starting point for the proofs of Theorems 2.3 and 2.4.

5. Proof of Theorem 2.3

We shall first prove Theorem 2.3. Of course, we can take the neighborhood U of the form $U=I\times B$ as before, satisfying all the properties that led to the proof of Proposition 4.1. We shall first prove Theorem 2.3 assuming that $u\in C^1(U)$. We assume that $f_j\doteq\mathrm{L}_j u\in C^\infty(U)$, for $j=1,\dots,n$, and estimate the term $u^+_{2,k,\varepsilon}(x,t)$. It is well known that we can find $F_j\in C^\infty(\mathbb{C}\times B_1)$, $j=1,\dots,n$, solving the approximate Cauchy problems

$$\begin{cases} F_j(Z(x,t),t) = f_j(x,t); & x \in J, \quad t \in B_1, \\ |(\bar{\partial}_z F_j)(Z(t,x)+iv,t)| \le C_k|v|^k, & |v| < \rho, \quad k \in \mathbb{Z}_+. \end{cases}$$

We have

$$\begin{aligned} &u_{2,k,\varepsilon}(x,t) = \\ &= \frac{1}{2\pi}\int_\gamma \sum_{j=1}^n \mathrm{d}s_j \left\{ \iint_{\mathbb{R}\times\mathbb{R}_+} (i\xi)^k e^{i\xi(Z(x,t)-Z(y,s))-\varepsilon\xi^2} \chi(y)\mathrm{L}_j u(y,s)\mathrm{d}Z(y,s)\mathrm{d}\xi \right\} = \\ &= \frac{1}{2\pi}\int_\gamma \sum_{j=1}^n \mathrm{d}s_j \left\{ \int_{\mathbb{R}_+}\int_J (i\xi)^k e^{i\xi(Z(x,t)-Z(y,s))-\varepsilon\xi^2} F_j(Z(y,s),s)\mathrm{d}Z(y,s)\mathrm{d}\xi \right\} \\ &\quad + \frac{1}{2\pi}\int_\gamma \sum_{j=1}^n \mathrm{d}s_j \left\{ \int_{\mathbb{R}_+}\int_{I\setminus J} (i\xi)^k e^{i\xi(Z(x,t)-Z(y,s))-\varepsilon\xi\cdot\xi} \chi(y)\mathrm{L}_j u(y,s)\mathrm{d}Z(y,s)\mathrm{d}\xi \right\}. \end{aligned}$$

Notice that for each $x \in J$, $t \in B_1 \setminus \Sigma_x$ and $s \in \gamma$ all fixed, the y-integration is indeed the integral of a one form in z over $\{Z(y,s) : s \in J\}$. Using the extension of f_j to the z-space and performing the change in the ξ-integral via

$$\zeta = \xi + \frac{i}{2}\frac{x - \operatorname{Re} z}{|x - \operatorname{Re} z|}\xi$$

we have the right to apply Stokes' Theorem in the following way:

$$\begin{aligned} &u_{2,k,\varepsilon}(x,t) = \\ &= \frac{1}{2\pi}\int_\gamma \sum_{j=1}^n \mathrm{d}s_j \left\{ \int_{\mathbb{R}_+}\int_J (i\xi)^k e^{i\xi(Z(x,t)-Z(y,s)-i\sigma)-\varepsilon\xi^2} F_j(Z(y,s)+i\sigma,s)\mathrm{d}Z(y,s)\mathrm{d}\xi \right\} \\ &\quad - \frac{1}{2\pi}\int_\gamma \sum_{j=1}^n \mathrm{d}s_j \left\{ \int_{\mathbb{R}_+}\int_{Z(J,s)+i[0,\sigma]} (i\xi)^k e^{i\xi(Z(x,t)-z)-\varepsilon\xi^2} \frac{\partial}{\partial \overline{z}} F_j(z,s)\mathrm{d}\overline{z}\wedge \mathrm{d}z\mathrm{d}\xi \right\} \\ &\quad + \frac{1}{2\pi}\int_\gamma \sum_{j=1}^n \mathrm{d}s_j \left\{ \int_{\mathbb{R}_+}\int_{\partial Z(J,s)+i[0,\sigma]} (i\xi)^k e^{i\xi(Z(x,t)-z)-\varepsilon\xi^2} F_j(z,s)\mathrm{d}z\mathrm{d}\xi \right\} \\ &\quad + \frac{1}{2\pi}\int_\gamma \sum_{j=1}^n \mathrm{d}s_j \left\{ \int_{\mathbb{R}_+}\int_{I\setminus J} (i\xi)^k e^{i\xi(Z(x,t)-Z(y,s))-\varepsilon\xi^2} \chi(y)\mathrm{L}_j u(y,s)\mathrm{d}Z(y,s)\mathrm{d}\xi \right\} \\ &= \frac{1}{2\pi}\int_\gamma \sum_{j=1}^n \mathrm{d}s_j \left\{ \int_J \int_{\zeta(\mathbb{R}_+)} (i\zeta)^k e^{i\zeta(Z(x,t)-Z(y,s)-i\sigma)-\varepsilon\zeta^2} F_j(Z(y,s)+i\sigma,s)\mathrm{d}\zeta\,\mathrm{d}Z(y,s) \right\} \end{aligned}$$

$$-\frac{1}{2\pi}\int_{\gamma}\sum_{j=1}^{n}\mathrm{d}s_j\left\{\int_{Z(J,s)+i[0,\sigma]}\int_{\zeta(\mathbb{R}_+)}(i\zeta)^k e^{i\zeta(Z(x,t)-z)-\varepsilon\zeta^2}\frac{\partial}{\partial\overline{z}}F_j(z,s)\mathrm{d}\zeta\,\mathrm{d}\overline{z}\wedge\mathrm{d}z\right\}$$

$$+\frac{1}{2\pi}\int_{\gamma}\sum_{j=1}^{n}\mathrm{d}s_j\left\{\int_{\partial Z(J,s)+i[0,\sigma]}\int_{\zeta(\mathbb{R}_+)}(i\zeta)^k e^{i\zeta(Z(x,t)-z)-\varepsilon\zeta^2}F_j(z,s)\mathrm{d}\zeta\,\mathrm{d}z\right\}$$

$$+\frac{1}{2\pi}\int_{\gamma}\sum_{j=1}^{n}\mathrm{d}s_j\left\{\int_{I\setminus J}\int_{\zeta(\mathbb{R}_+)}(i\zeta)^k e^{i\zeta(Z(x,t)-Z(y,s))-\varepsilon\zeta^2}\chi(y)\mathrm{L}_j u(y,s)\mathrm{d}\zeta\,\mathrm{d}Z(y,s)\right\}$$

Here $\sigma>0$ is sufficiently small. In view of (11), if $0\le\lambda\le\sigma$ and $s=\gamma(\tau)$ then

$$\begin{aligned}\mathrm{Re}\,\{i\zeta\cdot(Z(x,t)-Z(y,t)-i\lambda)-\varepsilon\zeta\cdot\zeta\}\le&-\frac{1}{2}\big((1-\theta)(8A)^{-1}\big)^{\frac{1}{1-\theta}}\tau^{\frac{1}{1-\theta}}\xi-\frac{3\varepsilon}{4}\xi^2\\&-\frac{1}{4}|x-y|\xi-\lambda\xi\end{aligned}$$

In the first, third and fourth integrals we use that the exponentials are bounded by $e^{-\sigma\xi}$, $e^{-\frac{d}{8}\xi}$ and $e^{-\frac{d}{8}\xi}$, respectively, if $x\in J/2$. Thus using the same argument as before one can estimate the absolute value of the first, third and fourth integrals by $C^{k+1}k!$, for some positive constant C (when $x\in J/2$, $t\in B_2\setminus\Sigma_x$). Now in the second integral we use the estimate of $|\bar\partial F_j|$ to obtain

$$\left|\int_{Z(I_1,s)+i[0,\sigma]}\int_{\zeta(\mathbb{R}_+)}(i\zeta)^k e^{i\zeta(Z(x,t)-z)-\varepsilon\zeta\cdot\zeta}\frac{\partial}{\partial\overline{z}}F_j(z,s)\mathrm{d}\zeta\,\mathrm{d}\overline{z}\wedge\mathrm{d}z\right|\le$$

$$\le\mathrm{Const.}C_k\int_0^{\sigma}\int_0^{+\infty}\xi^k e^{-\lambda\xi}\lambda^{k+1}\mathrm{d}\xi\,\mathrm{d}\lambda.$$

Taking account of Proposition 4.1 and this last inequality, and remembering again that Σ_x is a proper analytic subset of B_1, we conclude that all derivatives $\mathrm{M}^k u_\varepsilon^+$ are bounded in $J/2\times B_1$ by constants which are independent of $\varepsilon>0$. If we repeat the argument for u_ε^+ substituted by u_ε^-, we now conclude that all derivatives $\mathrm{M}^k u_\varepsilon$ are bounded in $J/2\times B_1$ by constants which are independent of $\varepsilon>0$. On the other hand, if $\alpha\in\mathbb{Z}_+^n$, $\alpha\neq 0$, and if $k\in\mathbb{Z}_+$ then

$$(\mathrm{M}^k\partial_t^\alpha u_\varepsilon)(x,t)=\frac{1}{2\pi}\int_{\mathbb{R}}\int_I(i\xi)^k\mathrm{L}^\alpha[u(y,t)\chi(y)]e^{i\xi(Z(x,t)-Z(y,t))-\varepsilon|\xi|^2}Z_y(y,t)\mathrm{d}y\mathrm{d}\xi$$

are bounded in $J/2\times B_1$, by constants which do not depend on ε. Here we have written $\mathrm{L}^\alpha=\mathrm{L}_1^{\alpha_1}\cdots\mathrm{L}_n^{\alpha_n}$ (of course the integrals which contain the terms $(\mathrm{L}^\alpha\chi)u$ occur outside J and hence can be treated as before).

Summing up, the family $\{u_\varepsilon\}_{\varepsilon>0}$ is bounded in the Montel space $C^\infty((J/2)\times B_1)$ and hence, for some sequence $\varepsilon_j\searrow 0$, $\{u_{\varepsilon_j}\}$ converges in $C^\infty((J/2)\times B_1)$. Since $u_\varepsilon\to u$ in $C^1(J\times B)$ we reach the conclusion that $u\in C^\infty((J/2)\times B_1)$. This concludes the proof when we assume that $u\in C^1(U)$.

We have now to prove the case when $u \in \mathcal{D}'(U)$. We first take $U_1 \subset\subset U$ an open set containing the origin and such that the second order operator

$$\mathsf{P} \doteq \mathrm{L}_1^2 + \cdots + \mathrm{L}_n^2 + \mathrm{M}^2$$

is elliptic in a neighborhood of the closure of U_1. This is indeed possible for P coincides with the Laplace operator at the origin. The main feature of P is that it commutes with all vector fields L_j, and hence with the operator d_0', which in our set up is identified to the operator

$$\mathrm{L}u = \sum_{j=1} \mathrm{L}_j u \, \mathrm{d}t_j$$

(P acts on forms by acting coefficientwise).

Since the restriction of u to U_1 is a distribution of finite order, we can find $m \in \mathbb{N}$ and $v \in C^1(U_1)$ such that $\mathsf{P}^m v = u$ in U_1. Now

$$\mathrm{L}u = \mathrm{L}\mathsf{P}^m v = \mathsf{P}^m \mathrm{L}v.$$

Since P^m is hypoelliptic it follows that $\mathrm{L}v$ is smooth and then, by what we have already proved, v (and hence u) is smooth in some neighborhood $U_0 \subset U_1$ of the origin (which can be taken independent of u).

The proof of Theorem 2.3 is now complete.

6. Proof of Theorem 2.4

The proof of Theorem 2.4 follows the same lines of that of Theorem 2.3. From Proposition 4.1 and the decomposition of the term $u_{2,\varepsilon}(x,t)$, we just have to estimate the family

$$I_\varepsilon^+(x,t) \doteq \frac{1}{2\pi} \int_\gamma \sum_{j=1}^n \mathrm{d}s_j \left\{ \int_{Z(J,s)+i[0,\sigma]} \int_{\zeta(\mathbb{R}_+)} (i\zeta)^k e^{i\zeta(Z(x,t)-z)-\varepsilon\zeta\cdot\zeta} \frac{\partial}{\partial \bar{z}} F_j(z,s) \mathrm{d}\zeta \, \mathrm{d}\bar{z} \wedge \mathrm{d}z \right\}.$$

For this, we now take into account the fact that $f_j = \mathrm{L}_j u \in G^s(U)$, $j = 1, \ldots, n$, and then solve the approximate Cauchy problems (cf. [1], Theorem 4.1),

$$\begin{cases} F_j(Z(x,t),t) = f_j(x,t); & x \in J, \quad t \in B_1, \\ |(\bar{\partial}_z F_j)(Z(t,x) + iv, t)| \le C^{k+1} k!^{s-1} |v|^k, & |v| < \rho, \quad k \in \mathbb{Z}_+, \end{cases}$$

with $F_j \in C^\infty(\mathbb{C} \times B_1)$, $j = 1, \ldots, n$.

Thus

$$\begin{aligned} \left|I_\varepsilon^+(x,t)\right| &\le \mathrm{Const} \cdot C^{k+2} (k+1)!^{s-1} \int_0^\sigma \int_0^{+\infty} \left(\frac{3}{2}\xi\right)^k e^{-\lambda\xi} \lambda^{k+1} \mathrm{d}\xi \mathrm{d}\lambda \\ &\le \mathrm{Const} \cdot C^{k+2} \left(\frac{3}{2}\right)^k (k+1)!^{s-1} \int_0^\sigma \int_0^{+\infty} \rho^k e^{-\rho} \mathrm{d}\rho \mathrm{d}\lambda \end{aligned}$$

$$\leq C_1^{k+1} k!^s.$$

This property, together with the fact that $\mathrm{L}^\alpha \mathrm{M}^k u$ satisfy the Gevrey estimates of order s if $\alpha \neq 0$ (recall that $\mathrm{L}_1, \ldots, \mathrm{L}_n, \mathrm{M}$ are real analytic and span $\mathbb{C} \otimes TU$), show that $u \in G^s((J/2) \times B_1)$.

7. An extension of Theorem 2.4

The advantage of using an almost analytic extension is that there is no need to estimate the derivatives of the cut-off function χ, and so our approach can be applied to the regular Denjoy-Carleman set up (regular in the sense of Dyn'kin, see [7] and [1]). More precisely, a sequence of non-negative numbers $\mathcal{M} = (M_k)_{k \in \mathbb{N}}$ is *regular* if, after defining $m_k = M_k / k!$, the following properties hold:

- $m_0 = m_1 = 1$;
- $m_k^2 \leq m_{k-1} \cdot m_{k+1}$, $k \geq 1$;
- $\sup_k (m_{k+1}/m_k) < \infty$;
- $\lim_k m_k^{1/k} = \infty$.

In the statement below we shall denote by $C^{\mathcal{M}}(U)$ the space of all smooth functions on U for which, given any compact set K of U, there is a constant $C > 0$ such that

$$\sup_K |(\partial_t^\alpha \partial_x^k u)| \leq C^{|\alpha|+k+1} M_{|\alpha|+k}, \quad \alpha \in \mathbb{Z}_+^n,\ k \in \mathbb{Z}_+.$$

As in the proof of Theorem 2.4, we obtain an equivalent definition substituting $\mathrm{L}^\alpha \mathrm{M}^k$ for $\partial_t^\alpha \partial_x^k$. Notice also that $C^\omega(U)$ is contained in $C^{\mathcal{M}}(U)$.

Theorem 7.1. *Let T' define a real-analytic, corank one locally integrable structure on an open neighborhood Ω of the origin in $\mathbb{R}^{n+1}$. If T' is hypocomplex in Ω and if it satisfies condition (N) then given $U \subset \Omega$, an open neighborhood of the origin, there is another such neighborhood $V \subset U$ for which the following is true: if u is a distribution solution of (2) in U and if the coefficients of f belong to $C^{\mathcal{M}}(U)$ then $u \in C^{\mathcal{M}}(V)$. Here $\mathcal{M}$ denotes any regular Denjoy-Carleman sequence.*

The proof of this result is identical with that of Theorem 2.4, where now the approximate Cauchy problems that must be solved are

$$\begin{cases} F_j(Z(x,t),t) = f_j(x,t); & x \in J, \quad t \in B_1, \\ |(\bar{\partial}_z F_j)(Z(t,x) + iv, t)| \leq C^{k+1} m_k |v|^k, & |v| < \rho, \quad k \in \mathbb{Z}_+. \end{cases}$$

According to ([1], Theorem 4.1) these problems have solutions $F_j \in C^1(\mathbb{C} \times B_1)$, $j = 1, \ldots, n$, which is enough for the argument to work.

Data availability

No data was used for the research described in the article.

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